Cobb-douglas production function

Consider a firm that produces a good Q with two factors, capital, K, and labor, L. The production function is

$$Q = 2\sqrt{KL}$$

Let p be the price of the good Q, s the hourly wage, and c the cost of using one unit of capital.

(i) Calculate the associated cost function.

(ii) Calculate the labor demand when the capital stock is $K = 4$.

(iii) Calculate the optimal value of the profits at the prices $(p, s, c) = (2, 1, 2)$.

Solution

(i) **Cost Function.**

To calculate the cost function, we must solve the following problem:

$$\min_{K,L} \; sL + cK \quad \text{s.t.} \quad Q_0 = 2\sqrt{KL}$$

where Q_0 is a given level of production. The first-order conditions are

$$c - \lambda \frac{1}{2} L^{\frac{1}{2}} K^{-\frac{1}{2}} = 0$$
$$s - \lambda \frac{1}{2} K^{\frac{1}{2}} L^{-\frac{1}{2}} = 0$$
$$2\sqrt{KL} - Q_0 = 0$$

where λ represents the Lagrangian multiplier. Combining these equations, we obtain

$$\frac{2c}{L^{1/2}K^{-1/2}} = \frac{2s}{L^{-1/2}K^{1/2}}$$

$$K = \frac{s}{c} L$$

Substituting this into the last equation, we obtain

$$Q_0 = 2\sqrt{\frac{s}{c} LL}$$

$$L = \frac{1}{2} Q_0 \sqrt{\frac{c}{s}}$$

Finally, substituting this expression for L into the equation for K, we obtain

$$K = \frac{1}{2} Q_0 \sqrt{\frac{s}{c}}$$

We can calculate the cost function by substituting these expressions for L and K into the objective function, i.e.

$$C(s, c, Q_0) = sL + cK = s\left(\frac{1}{2} Q_0 \sqrt{\frac{c}{s}}\right) + c\left(\frac{1}{2} Q_0 \sqrt{\frac{s}{c}}\right) = Q_0\sqrt{sc}.$$

(ii) **Labor Demand**

To obtain the labor demand, we must start by stating the profit function:

$$\pi(Q, K, L; p, s, c) = pQ - sL - cK$$

We know that the production volume Q is determined by the production function. Also, we have assumed that the capital stock is given, $K = 4$. Therefore, the profit function becomes

$$\pi(L, p, s, c) = p2\sqrt{4L} - sL - 4c$$

We obtain the labor demand by solving the problem

$$\max_{L} \; p2\sqrt{4L} - sL - 4c$$

The first-order condition is

$$2pL^{-\frac{1}{2}} - s = 0$$

Solving for L, we obtain

$$L(p,s) = \left(\frac{2p}{s}\right)^2.$$

(iii) **Profits**

Substituting the price vector $(p,s,c) = (2,1,2)$ into the expression for L, we obtain $L = 16$. Therefore, the optimal level of profits is obtained by substituting the price vector and the labor demand at these prices into the profit function,

$$\pi = 4\sqrt{(4)(16)} - 16 - 8 = 8.$$